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GDDs WITH TWO ASSOCIATE CLASSES AND WITH THREE GROUPS OF SIZES $3, n$ AND n

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Abstract

A group divisible design $\mathsf{GDD}(v = 3+n+n, 3, 3, \lambda_1, \lambda_2)$ is an ordered pair $(V, \mathcal{B})$ where V is an $(3+n+n)$-set of symbols and $\mathcal{B}$ is a collection of 3-subsets (called blocks) of V satisfying the following properties: the $(3+n+n)$-set is divided into 3 groups of sizes 3, n and n; each pair of symbols from the same group occurs in exactly λ_1 blocks in $\mathcal{B}$; and each pair of symbols from different groups occurs in exactly λ_2 blocks in $\mathcal{B}$. Let λ_1, λ_2 be positive integers. Then the spectrum of λ_1, λ_2, denoted by $\mathsf{Spec}(\lambda_1, \lambda_2)$, is defined by

$$\mathsf{Spec}(\lambda_1, \lambda_2) = \{n \in \mathbb{N} : a\ \mathsf{GDD}(v = 3+n+n, 3, 3, \lambda_1, \lambda_2) \text{ exists}\}.$$

We find the spectrum $\mathsf{Spec}(\lambda_1, \lambda_2)$ for all $\lambda_1 \geq \lambda_2$.

1 Introduction

A *pairwise balanced design* is an ordered pair $(S, \mathcal{B})$, denoted $\mathsf{PBD}(S, \mathcal{B})$, where S is a finite set of symbols and $\mathcal{B}$ is a collection of subsets of S called *blocks*,

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such that each pair of distinct elements of S occurs together in exactly one block of $\mathcal{B}$. Here $|S| = v$ is called the *order* of the PBD. Note that there is no condition on the size of the blocks in $\mathcal{B}$. If all blocks are of the same size k, then we have a *Steiner system* $\mathrm{S}(v, k)$. A PBD with index λ can be defined similarly; each pair of distinct elements occurs in λ blocks. If all blocks are same size, say k, then we get a balanced incomplete block design $\mathsf{BIBD}(v, b, r, k, \lambda)$. In other words, a $\mathsf{BIBD}(v, b, r, k, \lambda)$ is a set S of v elements together with a collection of b k-subsets of S, called blocks, where each point occurs in r blocks and each pair of distinct elements occurs in exactly λ blocks (see [5], [6], [11], [12]).

Note that in a $\mathsf{BIBD}(v, b, r, k, \lambda)$, the parameters must satisfy the necessary conditions

1. $vr = bk$ and
2. $\lambda(v - 1) = r(k - 1)$.

With these conditions a $\mathsf{BIBD}(v, b, r, k, \lambda)$ is usually written as $\mathsf{BIBD}(v, k, \lambda)$.

A *group divisible design* $\mathsf{GDD}(v = v_1 + v_2 + \cdots + v_g, g, k, \lambda_1, \lambda_2)$ is a collection of k-subsets (called blocks) of a v-set of symbols, where the v-set is divided into g groups of sizes $v_1, v_2, \ldots, v_g$; each pair of symbols from the same group occurs in exactly λ_1 blocks; and each pair of symbols from different groups occurs in exactly λ_2 blocks. Elements occurring together in the same group are called *first associates*, and elements occurring in different groups are called *second associates*. If the indices λ_1 and λ_2 were equal, then the design would be a BIBD (see [4]). The existence of such GDDs has been of interest over the years, going back to at least the work of Bose and Shimamoto in 1952 who began classifying such designs [1]. More recently, much work has been done on the existence of such designs when $\lambda_1 = 0$ (see [3] for a summary), and the designs here are called partially balanced incomplete block designs (PBIBDs) of group divisible type in [3]. The existence question for $k = 3$ has been solved by Sarvate, Fu and Rodger (see [5], [6]) when all groups are the same size.

In this paper, we continue to focus on blocks of size 3, solving the problem when the required designs having three groups of unequal size, namely, we consider the problem of determining necessary conditions for an existence of $\mathsf{GDD}(v = n_1 + n_2 + n_3, 3, 3, \lambda_1, \lambda_2)$ and prove that the conditions are sufficient for some infinite families. Since we are dealing on GDDs with three groups and block size 3, we will use $\mathsf{GDD}(n_1, n_2, n_3; \lambda_1, \lambda_2)$ for $\mathsf{GDD}(v = n_1 + n_2 + n_3, 3, 3, \lambda_1, \lambda_2)$ from now on, and we refer to the blocks as triples. We denote $(X, Y, Z; \mathcal{B})$ for a $\mathsf{GDD}(n_1, n_2, n_3; \lambda_1, \lambda_2)$ if X, Y and Z are n_1-set, n_2-set and n_3-set, respectively. Chaiyasena, et al. [2] have written a paper in this direction. In particular, they have solved the existence of a $\mathsf{GDD}(n, 2, 1; \lambda_1, \lambda_2)$ for $n \in \{2, \ldots, 6\}$. In [7], necessary and sufficient coditions were found for $\mathsf{GDD}(1, 1, n; 1, \lambda)$. Moreover, Hurd and Sarvate [8] found the necessary and sufficient conditions for $\mathsf{GDD}(1, 1, n; \lambda, 1)$. Recently, the existence of a $\mathsf{GDD}(1,2,n;\lambda_1,\lambda_2)$ has been solved by Hurd and Sarvate [9] when $n \geq 2$ and $\lambda_1 > \lambda_2$. More recenty, Lapchinda, et al. found in [10] all ordered triples $(n, \lambda_1, \lambda_2)$ of positive integers, with $\lambda_1 \geq \lambda_2$, such that a $\mathsf{GDD}(1,n,n;\lambda_1,\lambda_2)$ exists. We continue to investigate in this paper all triples of positive integers $(n, \lambda_1, \lambda_2)$ in which a $\mathsf{GDD}(3,n,n;\lambda_1,\lambda_2)$ exists for $\lambda_1 \geq \lambda_2$. We will see that necessary conditions on the existence of $\mathsf{GDD}(3,n,n;\lambda_1,\lambda_2)$ can be easily obtained by describing it graphically as follows.

Let G and H be multigraphs. A *G-decomposition* of H is a partition of the edges of H such that each element of the partition induces a copy of G. We denote $G|H$ for a G-decomposition of H. Let λK_v denote the multigraph on v vertices in which each pair of distinct vertices is joined by λ edges. Let G_1 and G_2 be vertex disjoint graphs. Then $G_1 \vee_\lambda G_2$ is the graph obtained from the union of G_1 and G_2 and by joining each vertex in G_1 to each vertex in G_2 with λ edges. Thus the existence of a $\mathsf{GDD}(n_1,n_2,n_3;\lambda_1,\lambda_2)$ is easily seen to be equivalent to the existence of a K_3-decomposition of $\lambda_1 K_{n_1} \vee_{\lambda_2} \lambda_1 K_{n_2} \vee_{\lambda_2} \lambda_1 K_{n_3}$.

The graph $\lambda_1 K_{n_1} \vee_{\lambda_2} \lambda_1 K_{n_2} \vee_{\lambda_2} \lambda_1 K_{n_3}$ is of order $n_1+n_2+n_3$ and size $\lambda_1[\binom{n_1}{2}+\binom{n_2}{2}+\binom{n_3}{2}]+\lambda_2(n_1n_2+n_1n_3+n_2n_3)$. It contains n_1 vertices of degree $\lambda_1(n_1-1)+\lambda_2(n_2+n_3)$, n_2 vertices of degree $\lambda_1(n_2-1)+\lambda_2(n_1+n_3)$, and n_3 vertices of degree $\lambda_1(n_3-1)+\lambda_2(n_1+n_2)$. Thus the existence of a K_3-decomposition of $\lambda_1 K_{n_1} \vee_{\lambda_2} \lambda_1 K_{n_2} \vee_{\lambda_2} \lambda_1 K_{n_3}$ implies

1. $3 \mid \{\lambda_1[\binom{n_1}{2}+\binom{n_2}{2}+\binom{n_3}{2}]+\lambda_2(n_1n_2+n_1n_3+n_2n_3)\}$, and
2. $2 \mid [\lambda_1(n_1-1)+\lambda_2(n_2+n_3)])$, $2 \mid [\lambda_1(n_2-1)+\lambda_2(n_1+n_3)]$, and $2 \mid [\lambda_1(n_3-1)+\lambda_2(n_1+n_2)]$.

2 Preliminary Results

In this section, we will review some known results concerning triple designs that will be used in the sequel, most of which are taken from [11]. Also we will show some new results that are needed for proving the main theorem.

Theorem 2.1. *Let v be a positive integer. Then there exists a $\mathsf{BIBD}(v,3,1)$ if and only if $v \equiv 1$ or $3 \pmod 6$.*

A $\mathsf{BIBD}(v,3,1)$ is usually called *Steiner triple system* and is denoted by $\mathsf{STS}(v)$. Let $(V,\mathcal{B})$ be an $\mathsf{STS}(v)$ where V is a set of v elements. Then the number of blocks or triples is $b = |\mathcal{B}| = v(v-1)/6$.

The following results on existence of λ-fold triple systems are well known (see, e.g., [11]).

Theorem 2.2. *Let n be a positive integer. Then a $\mathsf{BIBD}(n,3,\lambda)$ exists if and only if λ and n are in one of the following cases:*

(a) $\lambda \equiv 0 \pmod 6$ *and* $n \neq 2$,

(b) $\lambda \equiv 1$ *or* $5 \pmod 6$ *and* $n \equiv 1$ *or* $3 \pmod 6$,

(c) $\lambda \equiv 2$ *or* $4 \pmod 6$ *and* $n \equiv 0$ *or* $1 \pmod 3$, *and*

(d) $\lambda \equiv 3 \pmod 6$ *and* n *is odd.*

The following notations will be used throughout the paper for our constructions.

1. Let V be a v-set. $\mathsf{BIBD}(V, 3, \lambda)$ can be defined as

$$\mathsf{BIBD}(V, 3, \lambda) = \{\mathcal{B} : (V, \mathcal{B}) \text{ is a } \mathsf{BIBD}(v, 3, \lambda)\}.$$

2. Let X, Y and Z be three pairwise disjoint sets of cardinality n_1, n_2 and n_3, respectively. We define $\mathsf{GDD}(X, Y, Z; \lambda_1, \lambda_2)$ as

$$\mathsf{GDD}(X, Y, Z; \lambda_1, \lambda_2) = \{\mathcal{B} : (X, Y, Z; \mathcal{B}) \text{ is a } \mathsf{GDD}(n_1, n_2, n_3; \lambda_1, \lambda_2)\}.$$

3. When we say that $\mathcal{B}$ is a *collection* of subsets (blocks) of a v-set V, $\mathcal{B}$ may contain repeated blocks. Thus "$\cup$" in our context will be used for the union of multisets.

4. Finally, if we have a set X, the cardinality of X is denoted by $|X|$.

3 Necessity

Let λ_1, λ_2 be positive integers. Then the *spectrum of* λ_1, λ_2, denoted by $\mathsf{Spec}(\lambda_1, \lambda_2)$, is defined by

$$\mathsf{Spec}(\lambda_1, \lambda_2) = \{n \in \mathbb{N} : a \ \mathsf{GDD}(3, n, n; \lambda_1, \lambda_2) \text{ exists}\}.$$

Thus $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$, it is necessary that n satisfy the following conditions.

$$3 \mid [\lambda_1 n(n-1) + \lambda_2 n^2] \quad (1)$$
$$2 \mid [\lambda_1 (n-1) + \lambda_2 (n+1)] \quad (2)$$

By solving the system of congruences (1) and (2) corresponding to a given pair of (λ_1, λ_2), we obtain the following necessary condition for which $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$.

Theorem 3.1. *If* $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$, *then* λ_1, λ_2 *and* n *are related in* mod 6 *as in the following table.*

λ_2 / λ_1	0	1	2	3	4	5
0	all n	3	0, 3	1, 3, 5	0, 2, 3, 5	3
1	1,3	0, 2, 3, 5	3	0, 1, 3, 4	3, 5	0, 3
2	0, 1, 3, 4	3	0, 2, 3, 5	1, 3	0,3	3, 5
3	1, 3, 5	0, 3	3	all n	3	0, 3
4	0, 1, 3, 4	3, 5	0, 3	1, 3	0, 2, 3, 5	3
5	1, 3	0, 3	3, 5	0, 1, 3, 4	3	0, 2, 3, 5

The definition of $\mathsf{GDD}(3, n, n; \lambda_1, \lambda_2)$ along with the existence of $\mathsf{BIBD}(n, 3, 6)$ for all $n \geq 3$ if $\mathsf{GDD}(3, n, n; \lambda_1, \lambda_2)$ exists and $n \geq 3$, then for any positive integer i, $\mathsf{GDD}(3, n, n; \lambda_1 + 6i, \lambda_2)$ exists. This means that λ_1 can be arbitrary large.

4 Sufficiency

We prove in this section that the necessary conditions given in Theorem 3.1 become sufficient by constructing $\mathsf{GDD}(3, n, n; \lambda_1, \lambda_2)$ correspond to (λ_1, λ_2) given in the table. As we will construct $\mathsf{GDD}(3, n, n; \lambda_1, \lambda_2)$, we will use in this section X, Y, Z for sets of sizes $3, n, n$, respectively. The following observations are useful.

1. $\mathsf{GDD}(3, n, n; \lambda, \lambda)$ exists if and only if $\mathsf{BIBD}(2n + 3, 3, \lambda)$ exists.

2. $\mathsf{Spec}(\lambda, \lambda)$ can be obtained by applying results of Theorem 2.2 and we can characterize $\mathsf{Spec}(\lambda, \lambda)$ according to $\lambda \pmod 6$ as

   (a) Since $2n + 3$ is odd, it follows that $n \in \mathsf{Spec}(\lambda, \lambda)$ for all $\lambda \equiv 0$ or $3 \pmod 6$.

   (b) If $\lambda \equiv 1, 2, 4$ or $5 \pmod 6$, then $n \in \mathsf{Spec}(\lambda, \lambda)$ if and only if $n \equiv 0$ or $1 \pmod 3$.

3. Let $\langle X, Y, Z; \mathcal{B}\rangle$ be a $\mathsf{GDD}(3, n, n; \lambda_1, \lambda_2)$. Then for each positive integer i, $\langle X, Y, Z; i\mathcal{B}\rangle$ is a $\mathsf{GDD}(3, n, n; i\lambda_1, i\lambda_2)$, where $i\mathcal{B}$ is the union of i copies of $\mathcal{B}$. Thus, if $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$, then $n \in \mathsf{Spec}(i\lambda_1, i\lambda_2)$.

4. If $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$ and for each pair of non-negative integers (i, j) with $i \geq j$, then $n \in \mathsf{Spec}(\lambda_1 + 6i, \lambda_2 + 6j)$.

5. If a $\mathsf{BIBD}(2n + 3, 3, \lambda_1)$ exists and a $\mathsf{BIBD}(2n + 3, 3, \lambda_2)$ exists, then a $\mathsf{GDD}(3, n, n; \lambda_1 + \lambda_2, \lambda_2)$ exists.

With these observations and Theorem 3.1 we have the following results.

Theorem 4.1. *Let λ_1 and λ_2 be positive integers such that $\lambda_1 \geq \lambda_2$ and $\lambda_1 \equiv \lambda_2 \pmod 6$. Then, for all $n \geq 3$, $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$ if and only if $\lambda_1 \equiv 0, 1, 2, 3, 4$ or $5 \pmod 6$.*

Theorem 4.1 confirms that all entries in the main diagonal of the table are sufficient.

Theorem 4.2. *Let λ_1 and λ_2 be positive integers such that $\lambda_1 \geq \lambda_2$. If $n \equiv 3 \pmod 6$, then $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$.*

Proof. We want to show that the necessary conditions for $n \equiv 3 \pmod 6$ appearing in every entry of the table become sufficient.

Since $n \equiv 3 \pmod 6$, it follows that $2n+3 \equiv 3 \pmod 6$ and hence $\mathsf{BIBD}(2n+3, 3, i)$, $\mathsf{BIBD}(n, 3, i)$ and $\mathsf{BIBD}(3, 3, i)$ exist for all $i = 1, 2, 3, 4$, or 5. Thus, it is clear that if $\mathsf{GDD}(3, n, n; \lambda_1, \lambda_2)$ exists, then $\mathsf{GDD}(3, n, n; \lambda_1 + i, \lambda_2 + i)$ and $\mathsf{GDD}(3, n, n; \lambda_1 + i, \lambda_2)$ exist for all $i = 1, 2, 3, 4$, or 5.
We use

$$(a, b) \Rightarrow (a+1, b)$$

if $\mathsf{GDD}(3, n, n; a, b)$ exists, then $\mathsf{GDD}(3, n, n; a+1, b)$ exists and we use

$$\begin{array}{c}(a, b)\\ \Downarrow \\ (a+1, b+1)\end{array}$$

if $\mathsf{GDD}(3, n, n; a, b)$ exists, then $\mathsf{GDD}(3, n, n; a+1, b+1)$ exists. The following diagram shows that if $n \equiv 3 \pmod 6$, then $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$ for all (λ_1, λ_2) and n which are related in the table.

$$\begin{array}{ccccccccc}
(2,1) & \Rightarrow & (3,1) & \Rightarrow & (4,1) & \Rightarrow & (5,1) & \Rightarrow & (6,1)\\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow\\
(3,2) & \Rightarrow & (4,2) & \Rightarrow & (5,2) & \Rightarrow & (6,2) & \Rightarrow & (7,2)\\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow\\
(4,3) & \Rightarrow & (5,3) & \Rightarrow & (6,3) & \Rightarrow & (7,3) & \Rightarrow & (8,3)\\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow\\
(5,4) & \Rightarrow & (6,4) & \Rightarrow & (7,4) & \Rightarrow & (8,4) & \Rightarrow & (9,4)\\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow\\
(6,5) & \Rightarrow & (7,5) & \Rightarrow & (8,5) & \Rightarrow & (9,5) & \Rightarrow & (10,5))\\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow\\
(7,6) & \Rightarrow & (8,6) & \Rightarrow & (9,6) & \Rightarrow & (10,6) & \Rightarrow & (11,6)
\end{array}$$

□

Theorem 4.3. *Let λ_1 and λ_2 be positive integers such that $\lambda_1 \geq \lambda_2$. If $n \equiv 1 \pmod 6$, then $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$.*

Proof. We want to show that the necessary conditions for $n \equiv 1 \pmod 6$ appearing in every entry of the table become sufficient.

Since $n \equiv 1 \pmod 6$, it follows that $2n + 3 \equiv 5 \pmod 6$ and hence $\mathsf{BIBD}(2n+3, 3, 3)$, $\mathsf{BIBD}(n, 3, 1)$ and $\mathsf{BIBD}(3, 3, 1)$ exist. Thus, it is clear that if $\mathsf{GDD}(3, n, n; \lambda_1, \lambda_2)$ exists, then $\mathsf{GDD}(3, n, n; \lambda_1+3, \lambda_2+3)$ and $\mathsf{GDD}(3, n, n; \lambda_1+1, \lambda_2)$ exist.
We use

$$(a, b) \Rightarrow (a + 1, b)$$

if $\mathsf{GDD}(3, n, n; a, b)$ exists, then $\mathsf{GDD}(3, n, n; a + 1, b)$ exists and we use

$$\begin{array}{c}(a, b)\\ \Downarrow \\ (a + 3, b + 3)\end{array}$$

if $\mathsf{GDD}(3, n, n; a, b)$ exists, then $\mathsf{GDD}(3, n, n; a + 3, b + 3)$ exists. The following diagram shows that if $n \equiv 1 \pmod 6$, then $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$ for all (λ_1, λ_2) and n which are related in the table.

$$\begin{array}{ccccccccc}(4,3) & \Rightarrow & (5,3) & \Rightarrow & (6,3) & \Rightarrow & (7,3) & \Rightarrow & (8,3)\\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow\\ (7,6) & \Rightarrow & (8,6) & \Rightarrow & (9,6) & \Rightarrow & (10,6) & \Rightarrow & (10,6)\end{array}$$

□

Theorem 4.4. *Let λ_1 and λ_2 be positive integers such that $\lambda_1 \geq \lambda_2$. If $n \equiv 5 \pmod 6$, then $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$.*

Proof. We want to show that the necessary conditions for $n \equiv 5 \pmod 6$ appearing in every entry of the table become sufficient.

Since $n \equiv 5 \pmod 6$, it follows that $2n+3 \equiv 1 \pmod 6$ and hence $\mathsf{BIBD}(2n+3, 3, 1)$, $\mathsf{BIBD}(2n+3, 3, 3)$, $\mathsf{BIBD}(n, 3, 3)$ and $\mathsf{BIBD}(3, 3, 3)$ exist. Thus, it is clear that if $\mathsf{GDD}(3, n, n; \lambda_1, \lambda_2)$ exists, then $\mathsf{GDD}(3, n, n; \lambda_1+1, \lambda_2+1)$, $\mathsf{GDD}(3, n, n; \lambda_1+3, \lambda_2 + 3)$ and $\mathsf{GDD}(3, n, n; \lambda_1 + 3, \lambda_2)$ exist.
We use

$$(a, b) \Rightarrow (a + 1, b + 1)$$

if $\mathsf{GDD}(3, n, n; a, b)$ exists, then $\mathsf{GDD}(3, n, n; a + 1, b + 1)$ exists and we use

$$\begin{array}{c}(a, b)\\ \Downarrow \\ (a + 3, b + 3)\end{array}$$

if $\mathsf{GDD}(3, n, n; a, b)$ exists, then $\mathsf{GDD}(3, n, n; a+3, b+3)$ exists. The following diagram shows that if $n \equiv 5 \pmod 6$, then $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$ for all (λ_1, λ_2) and n which are related in the table.

$$\begin{array}{ccccc} (4,1) & \Rightarrow & (5,2) & \Rightarrow & (6,3) \\ \Downarrow & & \Downarrow & & \Downarrow \\ (7,4) & \Rightarrow & (8,5) & \Rightarrow & (9,6) \end{array}$$

□

Theorem 4.5. *Let λ_1 and λ_2 be positive integers such that $\lambda_1 \geq \lambda_2$. If $n \equiv 0$ or $4 \pmod 6$, then $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$.*

Proof. We want to show that the necessary conditions for $n \equiv 0$ or $4 \pmod 6$ appearing in every entry of the table become sufficient.

Since $n \equiv 0$ or $4 \pmod 6$, it follows that $2n + 3 \equiv 3$ or $5 \pmod 6$ and hence $\mathsf{BIBD}(2n+3, 3, 3), \mathsf{BIBD}(n, 3, 2)$ and $\mathsf{BIBD}(3, 3, 2)$ exist. Thus, it is clear that if $\mathsf{GDD}(3, n, n; \lambda_1, \lambda_2)$ exists, then $\mathsf{GDD}(3, n, n; \lambda_1+3, \lambda_2+3)$ and $\mathsf{GDD}(3, n, n; \lambda_1+2, \lambda_2)$ exist.
We use

$$(a, b) \Rightarrow (a+2, b)$$

if $\mathsf{GDD}(3, n, n; a, b)$ exists, then $\mathsf{GDD}(3, n, n; a+2, b)$ exists and we use

$$\begin{array}{c} (a, b) \\ \Downarrow \\ (a+3, b+3) \end{array}$$

if $\mathsf{GDD}(3, n, n; a, b)$ exists, then $\mathsf{GDD}(3, n, n; a+3, b+3)$ exists. The following diagram shows that if $n \equiv 0$ or $4 \pmod 6$, then $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$ for all (λ_1, λ_2) and n which are related in the table.

$$\begin{array}{ccc} (5,3) & \Rightarrow & (7,3) \\ \Downarrow & & \Downarrow \\ (8,6) & \Rightarrow & (10,6) \end{array}$$

□

Combining results in this section we obtain the following main theorem.

Theorem 4.6. *Let λ_1 and λ_2 be positive integers with $\lambda_1 \geq \lambda_2$ and n be an integer $n \geq 3$. Then $n \in \mathsf{Spec}(\lambda_1,\ \lambda_2)$ if and only if*

1. $3 \mid [\lambda_1 n(n-1) + \lambda_2 n^2]$, *and*

2. $2 \mid [\lambda_1(n-1) + \lambda_2(n+1)]$.

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